Northern Beaches Secondary College

Manly Selective Campus

2021 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. If $\omega = \text{cis}\left(\frac{\pi}{6}\right)$, then $\text{Im}(\omega^4 + 1)$ is:

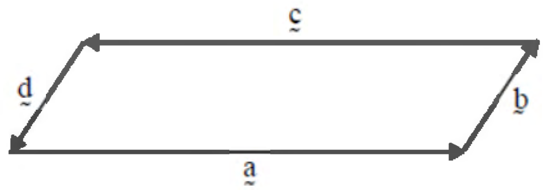
A. $\frac{\sqrt{3}}{2}$

B. $-\frac{1}{2}$

C. $1 + \frac{\sqrt{3}}{2}$

D. $\frac{1}{2}$

2. In the parallelogram $|\underset{\sim}{a}| = 2|\underset{\sim}{b}|$



Which one of the following statements is true?

A. $\underset{\sim}{a} = 2\underset{\sim}{b}$

B. $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

C. $\underset{\sim}{a} + \underset{\sim}{c} = 0$

D. $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

3. Which expression is equivalent to $\int (\ln x)^2 \, dx$?

A. $2\int (\ln x) \, dx$

B. $(\ln x)^2 - 2\int (\ln x) \, dx$

C. $x(\ln x)^2 - 2\int x(\ln x) \, dx$

D. $x(\ln x)^2 - 2\int (\ln x) \, dx$

4. Consider the statement:

P : for every $x \in \mathbb{Z}$ there exists $y \in \mathbb{Z}$ such that $y^2 > x$.

Which of the following statements is $\sim P$?

A. $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \quad y^2 > x$

B. $\exists x \in \mathbb{Z} \; \forall y \in \mathbb{Z} \quad y^2 \leq x$

C. $\forall x \in \mathbb{Z} \; \exists y \in \mathbb{Z} \quad y^2 > x$

D. $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, \quad y^2 \leq x$

5. If the vectors $\underset{\sim}{a} = m\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = m\underset{\sim}{i} + m\underset{\sim}{j} - 4\underset{\sim}{k}$ are perpendicular, then:

A. $m = -6$ or $m = 2$

B. $m = -2$ or $m = 6$

C. $m = -2$ or $m = 0$

D. $m = -1$ or $m = 1$

6. Without evaluating the integrals, which of the following integrals has the largest value?

A. $\displaystyle\int_{-\pi}^{\pi} x \cos x \, dx$

B. $\displaystyle\int_{-1}^{1} \ln(x^2 + 1) \, dx$

C. $\displaystyle\int_{0}^{1} (2^{-x} - 1) \, dx$

D. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sin^{-1} x\right)^3 dx$

7. A quadratic equation with positive real coefficients has complex roots α, β .

The value of $\arg \alpha + \arg \beta$ could be

A. π

B. $\dfrac{\pi}{2}$

C. 0

D. $\dfrac{3\pi}{2}$

8. A particle is moving in simple harmonic motion with an amplitude of 3 metres. Its speed is 4 m/s when the particle is 1 metre from the centre of motion.
What is the period of the motion?

A. $\sqrt{3}\,\pi$

B. $\sqrt{2}\,\pi$

C. $\dfrac{\pi}{\sqrt{3}}$

D. $\dfrac{\pi}{\sqrt{2}}$

9. Consider the following proof:

Proof: Assume a and b are odd integers. Then $a = 2c + 1$ and $b = 2d + 1$ for some $c, d \in \mathbb{Z}^{+}$.

Then $ab^2 = (2c + 1)(2d + 1)^2$
$= 8cd^2 + 8cd + 2c + 4d^2 + 4d + 1$
$= 2(4cd^2 + 4cd + c + 2d^2 + 2d) + 1$

Since $4cd^2 + 4cd + c + 2d^2 + 2d \in \mathbb{Z}$ we conclude that ab^2 is odd.

Which of the following statements is being proved?

A. If ab^2 is even, then a and b are even.

B. If ab^2 is even, then a is even or b is even.

C. If a and b are even, then ab^2 is even.

D. If a or b are even, then ab^2 is even.

10. Let the complex number z satisfy the equation $|z + 4i| = 3$. What are the greatest and least values of $|z + 3|$?

A. 8 and 5

B. 5 and 2

C. 8 and 3

D. 8 and 2

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions is Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new Writing Booklet.

a) What is the unit vector that has the same direction as $\underset{\sim}{v} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix}$? **1**

b) Let $z = \dfrac{1+i}{1-i}$ and $w = \dfrac{\sqrt{2}}{1-i}$

(i) Write each of z and w in modulus-argument form. **2**

(ii) On the same Argand diagram, sketch the points z, w and $z + w$ as poisition vectors. **2**

(iii) Deduce the exact value of $\tan\left(\dfrac{3\pi}{8}\right)$. **2**

c) On the Argand diagram, shade the region where both $|z - 1 - i| < 1$ and $0 \leq \arg z < \dfrac{\pi}{4}$ hold simualtenously . **2**

d) Find $\displaystyle\int \cos^3 x \, dx$ **2**

e) The velocity v m/s of a particle moving in simple harmonic motion along the x-axis is given by:

$$v^2 = 8 - 2x - x^2 \quad \text{, where } x \text{ is in metres.}$$

(i) Find the acceleration of the particle in terms of x. **1**

(ii) Find the centre and the amplitude of the motion. **2**

(iii) What is the maximum speed of the particle? **1**

Question 12 (16 marks) Start a new Writing Booklet.

a) Find the centre and radius of the sphere with equation. **2**

$$x^2 + y^2 + z^2 - x + 2y + 3z = 1$$

b) Prove by induction that $\dfrac{1}{n!} < \dfrac{1}{2^{n-1}}$ for $n \geq 3, n \in \mathbb{Z}^+$ **3**

c) The cubic polynomial $P(x) = 2(x-4)^3 + 16$ has one real solution.
Solve $P(x) = 0$ **3**

d) Prove that, for complex numbers z, w **2**

$$|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$$

e) Given that $|\underset{\sim}{a}| = 3$, $|\underset{\sim}{b}| = 2$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 4$, calculate the length of $2\underset{\sim}{a} - 3\underset{\sim}{b}$ **2**

f) A particle is moving along the x-axis. The square of its velocity is given by

$$v^2 = -2x^4 + 14x^2 - 24$$

where v m/s is the velocity and x is the displacement in metres from the origin. Initially the particle is stationary at $x = 2$.

(i) Show that **1**

$$a = -4x\left(x^2 - \frac{7}{2}\right)$$

where a is the acceleration of the particle.

(ii) Explain why the object starts moving in the negative direction. **1**

(iii) Where does the particle next come to rest? **2**

Question 13 (13 marks) Start a new Writing Booklet.

a) The equations of intersecting lines L and M are given below with respect to a fixed origin O. **2**

$$L : \underset{\sim}{r_1} = 11\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k} + \lambda\left(-2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}\right) \quad \text{and} \quad M : \underset{\sim}{r_2} = -5\underset{\sim}{i} + 11\underset{\sim}{j} + \underset{\sim}{k} + \mu\left(p\,\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

where λ and μ are parameters and p is a constant.
If L and M are perpendicular, what is the value of p?

b) Use the method of proof by contradiction to prove that the cube root of any prime number $p \geq 2$ is irrational. **3**

c) (i) Show that $f(x) = \dfrac{2 + x^2}{4 - x^2}$ can be written as $f(x) = -1 + \dfrac{6}{4 - x^2}$. **1**

(ii) Find the exact area enclosed by the graph of $y = f(x)$, the x-axis and the ordinates at $x = -1$ and $x = 1$. **3**

d) Prove by Mathematical Induction that the number of diagonals of a convex polygon with n vertices is $\dfrac{1}{2}n(n - 3)$, for $n \geq 4$. **3**

e) Find $\displaystyle\int \frac{x^3}{x^8 + 3}\,dx$ **2**

Question 14 (15 marks) Start a new Writing Booklet.

a) The points A and B have position vectors given by $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$.

(i) Find an expression for the vector $\overrightarrow{AB}$ in the form of $x\,\underset{\sim}{i} + y\,\underset{\sim}{j} + z\,\underset{\sim}{k}$. **1**

(ii) Show that the cosine of the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ is $\frac{4}{9}$. **2**

(iii) Hence, find the exact value of the area ΔOAB. **3**

b) By rewriting the equation in the form $a^2 + b^2 = 0$, or otherwise, disprove the statement: **2**

$$\exists\, x \in \mathbb{R} \text{ such that } x^6 + x^4 + 1 = 2x^2$$

c) (i) If $z = e^{i\theta}$, show that $z^n + z^{-n} = 2\cos(n\theta)$. **1**

(ii) Hence, or otherwise, determine the values of θ, where $0 \leq \theta < 2\pi$ such that: **3**

$$|e^{4i\theta} + 1| = \sqrt{3}$$

d) Find $\int e^{2\sqrt{x+1}}\,dx$ **3**

Question 15 (15 marks) Start a new Writing Booklet.

a) Let $I_n = \int_0^1 x^n \tan^{-1} x \, dx$ where $n = 0,1,2...$

(i) Show that $(n+1) I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} dx$ for $n \geq 0$. **2**

(ii) Hence, or otherwise, find the value of I_0 in simplest exact form. **2**

(iii) Show that $(n+3) I_{n+2} + (n+1) I_n = \frac{\pi}{2} - \frac{1}{n+2}$. **2**

(iv) Hence find the value of I_4. **2**

b) Assume that the tides rise and fall in simple harmonic motion. At low tide, the channel in a harbour is 8 metres deep, while at high tide the depth is 12 metres. Low tide occurs at 9 am and high tide occurs at 4 pm.

(i) What is the depth of the channel at 12:30 pm? **1**

(ii) What is the amplitude of the motion? **1**

(iii) Show that the depth y metres of the water in the channel is given by **3**

$$y = 10 - 2\cos\left(\frac{\pi}{7} t\right)$$

where t is the number of hours after low tide.

(iv) A ship needs at least 9 metres of water to pass through the channel safely. The ship takes one hour to traverse the length of the channel. Find the times between which the ship can navigate safely through the channel. **2**

Question 16 (15 marks) Start a new Writing Booklet.

a) Find:

(i) $$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\, dx$$ **3**

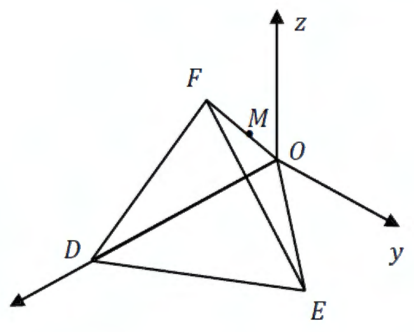
(ii) $$\int \frac{1}{4+5\cos x}\, dx \quad \text{using } t = \tan\frac{x}{2}$$ **3**

b) We are given that $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are three consecutive terms in a geometric series, where a, b and $c \in \mathbb{R}$.

(i) Show that $a^2 + c^2 \geq ab + bc$ **3**

(ii) Show that $\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{b^2}$ **2**

c) The faces of tetrahedron *ODEF* are comprised of equilateral triangles of side length 1 unit. Its base lies flat on the x-y plane with vertices at O, $D\,(1, 0, 0)$ and $E\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$ as shown.



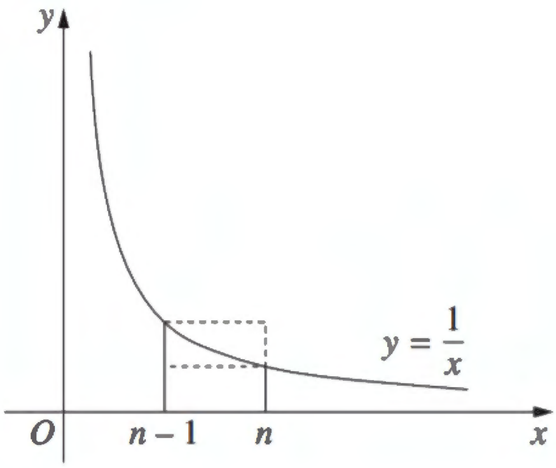
Prove the coordinates of M, the midpoint of FO, is $\left(\frac{1}{4}, \frac{\sqrt{3}}{12}, \frac{\sqrt{6}}{6}\right)$. **2**

Question 16 continues on page 13

Question 16 (continued)

d)

2



Let n be a positive integer greater than 1.

The area of the region under the curve $y = \dfrac{1}{x}$ from $x = n - 1$ to $x = n$ is between the areas of two rectangles, as shown in the diagram.

Show that

$$e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$$

End of paper

Ext2 Solutions

MC

1. A
2. C
3. D
4. B
5. A
6. B
7. C
8. B
9. D
10. D

11.

a) $\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{46}}(6i + 3j - k)$

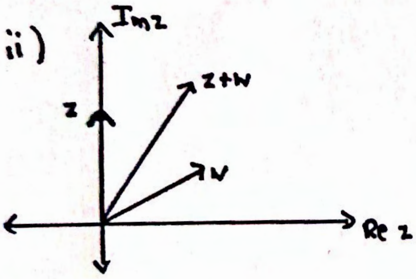
① mark for answer

b) i) $z = \text{cis}\,\frac{\pi}{2}$

$w = \text{cis}\,\frac{\pi}{4}$

① mark each

ii)



② marks all correct

① mark one error

0 marks two errors

iii) $\arg(z+w) = \frac{\arg z - \arg w}{2} + \arg w$

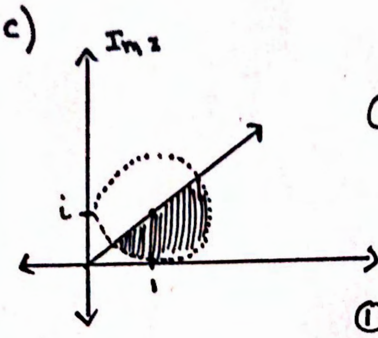
$= \frac{3\pi}{8}$ ① mark

$z+w = \frac{1}{\sqrt{2}} + i\left(1 + \frac{1}{\sqrt{2}}\right)$

$\tan\frac{3\pi}{8} = \frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

$= \sqrt{2} + 1$ ① mark for answer.

c)



① mark for correct circle and line.

① mark for correct shading.

d) $\int \cos^3 x \, dx$

$= \int (1 - \sin^2 x)\cos x \, dx$

$= \int (1 - u^2)\, du$ $\quad u = \sin x$, $du = \cos x \, dx$ ← ① mark

$= u - \frac{u^3}{3} + c$

$= \sin x - \frac{1}{3}\sin^3 x + c$ ① mark for answer

e) i) $\ddot{x} = -1 - x$ ① mark

ii) $\ddot{x} = -1(x+1)$

centre of motion $x = -1$ ① mark

when $v = 0$ $\quad x = -4$ and $x = 2$

$\therefore$ amplitude is 3 ① mark

iii) max speed occurs at centre of motion $x = -1$

$v^2 = 8 - 2(-1) - (-1)^2$

$v = \pm 3$ $\quad$ max speed 3 m/s ① mark for answer

Q12

a) $(x-\frac{1}{2})^2+(y+1)^2+(z+\frac{3}{2})^2=\frac{9}{2}$

centre $(\frac{1}{2}, -1, -\frac{3}{2})$,

radius $\frac{3}{\sqrt{2}}$ ← ① mark each

b) step1: Show it's true for $n=3$

LHS $=\frac{1}{6}$ RHS $=\frac{1}{4}$

$\frac{1}{6}<\frac{1}{4}$ true

step2: Assume $\frac{1}{k!}<\frac{1}{2^{k-1}}$ is true for $k\geq 3$ $k\in \mathbb{Z}^+$

step3: show $\frac{1}{(k+1)!}<\frac{1}{2^k}$ is true

LHS $=\frac{1}{(k+1)!}$ ① mark step 1

$=\frac{1}{k!(k+1)}$

$<\frac{1}{2^{k-1}(k+1)}$ ← ① mark up to here

$=\frac{2}{2^k(k+1)}$ ① mark final step

$=\frac{1}{2^k}\times\frac{2}{k+1}<\frac{1}{2^k}$

∴ by induction its true.

c) $x=2$ is the real solution

$P(x)=2x^3-24x^2+96x-112$

$=(x-2)(2x^2-20x+56)$

$=2(x-2)((x-5)^2+3)$

$=2(x-2)(x-5-\sqrt{3}i)(x-5+\sqrt{3}i)$

$=0$

when $x=2$ $x=5\pm\sqrt{3}i$

① mark each solution.

d) LHS $=(z-w)(\overline{z-w})+(z+w)(\overline{z+w})$

$=(z-w)(\bar{z}-\bar{w})+(z+w)(\bar{z}+\bar{w})$

$=z\bar{z}-z\bar{w}-\bar{z}w+z\bar{z}+\bar{z}w+w\bar{z}+\bar{w}w+w\bar{w}$

$=2|z|^2+2|w|^2$ ① mark for 1st line

$=2(|z|^2+|w|^2)$

$=$ RHS ① mark for rest of solution

e) $|2\underset{\sim}{a}-3\underset{\sim}{b}|^2=(2a-3b)\cdot(2a-3b)$

$=4a\cdot a-6a\cdot b-6a\cdot b+9b\cdot b$

$=4\times 9-12\times 4+9\times 4$

$=24$ ① mark working

$=2\sqrt{6}$ ① mark answer.

f) See alternate task solutions

Q13

a) $(-2i+j-4k)\cdot(pi+2j+2k)=0$

$-2p+2-8=0$ ① for working

$p=-3$ ① for answer

b) Assume $\sqrt[3]{p}$ is rational

∴ $\sqrt[3]{p}=\frac{a}{b}$ where a,b are $\mathbb{Z}^+$, $b\neq 0$ and a and b have no common factors.

$pb^3=a^3$

Thus pb^3 is divisible by a. But b is not divisible by a, so that means p is divisible by a

if $a = 1$ $p = \frac{1}{b^3}$

but $p \geq 2$ so not possible

if $a \geq 2$, $p \neq a$

since p is prime, p cannot be divisible by a

if $a \geq 2$ $p = a$ then $b^3 = a^3$

which contradicts our assumption

$\therefore$ $\sqrt[3]{p}$ is not rational

(1) mark assumption

(1) mark ~~working~~ explaining p is divisible by a.

(1) mark for working and concluding $\sqrt[3]{p}$ is not rational

c) i) $-1 + \frac{6}{4-x^2}$

$= \frac{-4+x^2+6}{4-x^2}$

$= \frac{x^2+2}{4-x^2}$ ① mark for working

ii) $\int_{-1}^{1} \left(\frac{6}{4-x^2} - 1\right) dx$

$= 2\int_0^1 \left(\frac{6}{4-x^2} - 1\right) dx$

① mark to here

$= 2\int_0^1 \left(\frac{\frac{3}{2}}{2-x} + \frac{\frac{3}{2}}{2+x} - 1\right) dx$

using partial fractions

① mark for integral

$= 2\left[-\frac{3}{2}\ln(2-x) + \frac{3}{2}\ln(2+x) - x\right]_0^1$

$= \left[3\ln\left(\frac{2+x}{2-x}\right) - 2x\right]_0^1$

$= 3\ln 3 - 2$ ← ① mark for answer

d) Step 1: $n=4$

$\frac{1}{2} \times 4 \times 1 = 2$ which is true

step 2: Assume the number of diagonals of a convex polygon with k vertices is $\frac{1}{2}k(k-3)$ for $k \geq 4$

Step 3: show for $k+1$ vertices the number of diagonals is $\frac{1}{2}(k+1)(k-2)$

n	4	5	6	7
diagonals	2	5	9	14

∴ when adding another vertice we always have to create two new lines, all diagonals remain in the k vertices convex polygon plus always $k-2$ new diagonals and the line used to create the new vertice. So for a $(k+1)$ vertice convex polygon we have

$\underbrace{\frac{1}{2}k(k-3)}_{k \text{ vertice}} + (k-2) + 1$ diagonals

$= \frac{1}{2}(k^2-3k) + k - 2 + 1$

$= \frac{1}{2}k^2 - \frac{1}{2}k - 1$

$= \frac{1}{2}(k^2 - k - 2)$

$= \frac{1}{2}(k-2)(k+1)$ as required

① mark for step 1

① mark for some explanation of how this was obtained

① mark for working in steps.

e) $= \int \frac{\frac{1}{4}\,du}{u^2+3}$ where $u = x^4$, $du = 4x^3$

$= \frac{1}{4\sqrt{3}} \tan^{-1}\left(\frac{x^4}{\sqrt{3}}\right) + C$

① for u-sub

① for answer

Q14

a) i) $\vec{AB} = -2i + j + 2k$ (1) mark

ii) $\dfrac{\vec{OA} \cdot \vec{AB}}{|\vec{OA}|\,|\vec{AB}|} = \cos\theta$ ← (1) mark

$\dfrac{-2+2+4}{\sqrt{9} \times \sqrt{9}} = \cos\theta$

$\cos\theta = \dfrac{4}{9}$ ← (1) mark for both

iii) $\sin\theta = \dfrac{\sqrt{65}}{9}$ ← (1) mark

$A = \frac{1}{2} \times \sqrt{9} \times \sqrt{9} \times \dfrac{\sqrt{65}}{9}$ ← (1) mark working

$= \frac{1}{2}\sqrt{65}$ units^2 ← (1) mark answer

b)

$x^6 + x^4 - 2x^2 + 1 = 0$

(1) mark → $(x^3)^2 + (x^2-1)^2 = 0$

$(x^3)^2 = -(x^2-1)^2$

but the square of any real number is always ≥ 0 ∴ so equation has no real solutions

(1) mark for disproving statement

[redacted] c) i) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= 2\cos n\theta$ (1) mark

ii) $|\cos 4\theta + 1 + i\sin 4\theta| = \sqrt{3}$

$3 = (\cos 4\theta + 1)^2 + \sin^2 4\theta$

$2\cos 4\theta + 2 = 3$ on expanding and simplifying

$\cos 4\theta = \frac{1}{2}$ $\quad 0 \leq 4\theta < 8\pi$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

(1) mark for [redacted]

(1) mark for $\cos 4\theta = \frac{1}{2}$

(1) mark for all solutions

d) $= \int e^{2\sqrt{u}}\, du$ $\quad$ where $u = x+1$, $du = dx$

$= \int 2w\, e^{2w}\, dw$ $\quad w = \sqrt{u}$, $dw = \dfrac{1}{2\sqrt{u}}\, du$

$= 2w \times \frac{1}{2} e^{2w} - \int \frac{1}{2} e^{2w} \times 2\, dw$ (by parts)

$= we^{2w} - \int e^{2w}\, dw$

$= we^{2w} - \frac{1}{2}e^{2w} + c$

$= \sqrt{x+1}\, e^{2\sqrt{x+1}} - \frac{1}{2}e^{2\sqrt{x+1}} + c$

(1) mark for

(1) mark for first line working in integration by parts

(1) mark for answer

Q15.

a) i) ii) iii) See alternate task solutions.

iv) $n=2 \quad 5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}$

$n=0 \quad 3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}$

but $I_0 = \frac{\pi}{4} - \ln\sqrt{2}$ → sub in and simplify

$\therefore 5I_4 + 3\left(\frac{\pi}{4} - \frac{1}{6} + \frac{1}{3}\ln\sqrt{2}\right) = \frac{\pi}{2} - \frac{1}{4}$

$I_4 = \frac{\pi}{20} + \frac{1}{20} - \frac{1}{5}\ln\sqrt{2} \doteq 0.13776$

① mark for ① mark for answer.

b) i) 10 metres ① mark

ii) amplitude is 2 ① mark

iii) period is 14 hrs

$\frac{2\pi}{n} = 14$ ① mark for period

$n = \frac{\pi}{7}$ ← ① for n

$a = 2$ centre is 10

$\therefore y = 10 - 2\cos\left(\frac{\pi}{7}t\right)$ ↙ ① for

iv) $-2\cos\left(\frac{\pi}{7}t\right) + 10 = 9$

$\cos\frac{\pi}{7}t = \frac{1}{2}$

$\frac{\pi}{3} = \frac{\pi}{7}t$

$t = 11.\dot{6}$ or $2.\dot{3}$

Ship can enter at 11:20 am and must exit any time before 7:40 pm

① for solving trig eq. ① for time

Q16

a) i)

$= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta} \times \frac{3}{2}\sec^2\theta \, d\theta$

where $x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta \, d\theta$

$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta} \, d\theta$

$= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta} \sin\theta \, d\theta$

$= -\frac{3}{16}\int_0^{1/2} \frac{1-u^2}{u^2} \, du$

where $u = \cos\theta \quad du = -\sin\theta \, d\theta$

$= \frac{3}{16}\left[u - \frac{1}{u}\right]_0^{1/2}$

$= \frac{3}{32}$

① for trig sub
① for u-sub
① for answer

ii) See alternate task solution

b) i) $\frac{1}{b} \div \frac{1}{a} = \frac{1}{c} \div \frac{1}{b}$

$ac = b^2$ ← ① mark

$(a-b)^2 \geq 0$

$a^2 + b^2 \geq 2ab$

$b^2 + c^2 \geq 2bc$

$a^2 + c^2 \geq 2ac$

$\therefore 2a^2 + 2b^2 + 2c^2 \geq 2(ab + ac + bc)$

$a^2 + b^2 + c^2 \geq ab + ac + bc$

but $ac = b^2$ ① mark

$\therefore a^2 + ac + c^2 \geq ab + ac + bc$

$a^2 + c^2 \geq ab + bc$

① for final working

ii) $\left(\frac{1}{a}-\frac{1}{c}\right)^2 \geq 0$

$\frac{1}{a^2}-\frac{2}{ac}+\frac{1}{c^2} \geq 0$

$\frac{1}{a^2}+\frac{1}{c^2} \geq \frac{2}{ac}$ ① for first line

but $ac = b^2$ ① for rest of solution

$\frac{1}{a^2}+\frac{1}{c^2} \geq \frac{2}{b^2}$

c) ΔMOD let $M = (a, b, c)$

$\vec{OM} \cdot \vec{OD} = |\vec{OM}|\,|\vec{OD}| \cos\frac{\pi}{3}$

$\frac{a\times 1 + b\times 0 + c\times 0}{\frac{1}{2}\times 1} = \cos\frac{\pi}{3}$

$a = \frac{1}{4}$

ΔMOE $\cos\frac{\pi}{3} = \frac{\vec{OM}\cdot\vec{OE}}{\frac{1}{2}\times 1}$

$\frac{1}{4} = \frac{a}{2} + \frac{\sqrt{3}}{2}b$

but $a = \frac{1}{4}$

$\frac{1}{4} = \frac{1}{8} + \frac{\sqrt{3}}{2}b$

$b = \frac{\sqrt{3}}{12}$ ① for a and b as same working

distance from O to M is

$\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{12}\right)^2 + c^2 = \left(\frac{1}{2}\right)^2$

$\frac{1}{16} + \frac{2}{144} + c^2 = \frac{1}{4}$ ① for point c

$c^2 = \frac{1}{6}$ $c > 0$

$\therefore c = \frac{\sqrt{6}}{6}$

$\therefore M\left(\frac{1}{4}, \frac{\sqrt{3}}{12}, \frac{\sqrt{6}}{6}\right)$

d) area of lower rectangle

$= \frac{1}{n}$

area of upper rectangle

$= \frac{1}{n-1}$

area under the curve

$= \int_{n-1}^{n} \frac{1}{x}\,dx$

$= \left[\ln x\right]_{n-1}^{n}$

$= \ln\frac{n}{n-1}$ ① mark

$\therefore \frac{1}{n} < \ln\frac{n}{n-1} < \frac{1}{n-1}$

$e^{\frac{1}{n}} < e^{\ln\frac{n}{n-1}} < e^{\frac{1}{n-1}}$

$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$

$e^{-1} > \left(\frac{n}{n-1}\right)^{-n} > e^{-\frac{n}{n-1}}$

$e^{-\frac{n}{n-1}} < \left(\frac{n-1}{n}\right)^n < e^{-1}$

$e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$

as required.

① mark for rest of working